EXERCISES [MAI 5.10-5.13]

BASIC INTEGRALS - AREAS

SOLUTIONS

Compiled by: Christos Nikolaidis

A. Paper 1 questions (SHORT)

INDEFINITE INTEGRALS

1.

$\int x^9 dx = \frac{x^{10}}{10} + c$
$\int 20x^9 dx = 20\frac{x^{10}}{10} + c = 2x^{10} + c$
$\int x^{-9} dx = \frac{x^{-8}}{-8} + c \quad \left(= -\frac{1}{8x^8} + c \right)$
$\int 16x^{-9} dx = 16\frac{x^{-8}}{-8} + c = -2x^{-8} + c \quad \left(= -\frac{2}{x^8} + c \right)$
$\int (x^3 + x^2 + x + 3) dx = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + 3x + c$
$\int (4x^3 - 12x^2 + 6x + 3) dx = x^4 - 4x^3 + 3x^2 + 3x + c$
$\int (x^{-3} + x^{-2} + 3) dx = \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} + 3x + c \quad \left(= -\frac{1}{2x^2} - \frac{1}{x} + 3x + c \right)$
$\int (4x^{-3} - 12x^{-2} + 3) dx = -2x^{-2} + 12x^{-1} + 3x + c = \frac{-2}{x^2} + \frac{12}{x} + 3x + c$
$\int e dx = ex + c$
$\int \pi^2 dx = \pi^2 x + c$

2.

$\int \left(\frac{4}{x^3} - \frac{12}{x^2} + 3 \right) dx = \int (4x^{-3} - 12x^{-2} + 3) dx = -2x^{-2} + 12x^{-1} + 3x + c = \ldots$
$\int \left(\frac{8}{x^5} - \frac{12}{x^3} + 6x \right) dx = \int (8x^{-5} - 12x^{-3} + 6x) dx \; -2x^{-4} + 6x^{-2} + 3x^2 + c = \frac{-2}{x^4} + \frac{6}{x^2} + 3x^2 + c$
$\int \left(\frac{x^3}{4} - \frac{2x^2}{3} - \frac{x}{7} + 3 \right) dx = \int \left(\frac{1}{4}x^3 - \frac{2}{3}x^2 - \frac{1}{7}x + 3 \right) dx = \frac{1}{16}x^4 - \frac{2}{9}x^3 - \frac{1}{14}x^2 + 3x + c$
$\int \left(\frac{1}{4x^3} - \frac{2}{3x^2} + 3 \right) dx = \int \left(\frac{1}{4}x^{-3} - \frac{2}{3}x^{-2} + 3 \right) dx = -\frac{1}{8}x^{-2} + \frac{2}{3}x^{-1} + 3x + c = \ldots$
$\int \left(\frac{3}{4x^3} - \frac{2x^2}{3} \right) dx = \int \left(\frac{3}{4}x^{-3} - \frac{2}{3}x^2 \right) dx = -\frac{3}{8}x^{-2} - \frac{2}{9}x^3 + c$
$\int (x+1)(x+2) dx = \int (x^2 + 2x + x + 2) dx = \int (x^2 + 3x + 2) dx = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$
$\int (2x^2 + 5x + 1)(3x - 4) dx = \int (6x^3 + 7x^2 - 17x - 4) dx = \frac{3x^4}{2} + \frac{7x^3}{3} - \frac{17x^2}{2} - 4x + c$

$\int \frac{2x^5+5x^3+x^2}{x^2}dx = = \int\left(2x^3+5x+1\right)dx = x^3+5\frac{x^2}{2}+x+c$
Z $\int \frac{2x^3+5x+1}{x^3}dx = \int\left(2+5x^{-2}+x^{-3}\right)dx = 2x-5x^{-1}-\frac{x^{-2}}{2}+c$
$\int \frac{2x^7+5x+4}{3x^3}dx = \int\left(\frac{2}{3}x^4+\frac{5}{3}x^{-2}+\frac{4}{3}x^{-3}\right)dx = \frac{2x^5}{15}-\frac{5}{3x}-\frac{2}{3x^2}+c$

3. $f(x) = 2x^3 + c$

$f(0) = 3 \Leftrightarrow c = 3$

$f(x) = 2x^3 + 3$

4. (a) $f'(x) = 4x^3 + c$

$f'(0) = 3 \Leftrightarrow c = 3$

$f'(x) = 4x^3 + 3$

(b) $f(x) = x^4 + 3x + d$

$f(0) = 2 \Leftrightarrow c = 2$

$f(x) = x^4 + 3x + 2$

5. $f'(x) = 1 - x^2$

$f(x) = \int (1 - x^2)\mathrm{d}x = x - \frac{x^3}{3} + C$

$f(3) = 0 \Rightarrow 3 - 9 + C = 0 \Rightarrow c = 6$

$f(x) = x - \frac{x^3}{3} + 6$

6. $y = \int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \frac{x^4}{4} + \frac{2x^2}{2} - x + c$

$13 = \frac{16}{4} + 4 - 2 + c \Rightarrow c = 7$

$y = \frac{x^4}{4} + x^2 - x - 7$

7. $y = x^3 - 5x + c$

substitute (2, 6) to find c $\left(6 = 2^3 - 5(2) + c\right) \Rightarrow c = 8$

$y = x^3 - 5x + 8$

8. $f'(x) = -2x + 3$

$f(x) = \frac{-2x^2}{2} + 3x + c$

$1 = -1 + 3 + c \Rightarrow c = -1$

$f(x) = -x^2 + 3x - 1$

DEFINITE INTEGRALS

9.

$\int_0^1 x^9 dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$
$\int_0^1 20x^9 dx = \left[2x^{10}\right]_0^1 = 2$
$\int_1^2 x^{-2} dx = \left[\frac{x^{-1}}{-1}\right]_1^2 = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2}+1=\frac{1}{2}$
$\int_1^2 8x^{-3} dx = \left[8\frac{x^{-2}}{-2}\right]_1^2 = \left[-\frac{4}{x^2}\right]_1^2 = -1+4=3$
$\int_0^2 6x^2 dx = \left[2x^3\right]_0^2 = 16-0=16$
$\int_1^2 6x^2 dx = \left[2x^3\right]_1^2 = 16-2=14$
$\int_0^1 (2x+3)dx = \left[x^2+3x\right]_0^1 = 4-0=4$
$\int_1^2 (2x+3)dx = \left[x^2+3x\right]_1^2 = 10-4=6$
$\int_0^2 (2x+3)dx = \left[x^2+3x\right]_1^2 = 10-0=10$
$\int_{-2}^2 (2x+3)dx = \left[x^2+3x\right]_{-2}^2 = 10-(-2)=12$
$\int_0^{10} xdx = \left[\frac{x^2}{2}\right]_0^{10} = 50$
$\int_0^{10} 5dx = \left[5x\right]_0^{10} = 50$
$\int_0^{10} dx = \left[x\right]_0^{10} = 10$
$\int_4^{10} dx = \left[x\right]_4^{10} = 10-4=6$
$\int_a^b dx = \left[x\right]_a^b = b-a$

10. $\int_1^a \left(3x^2+1\right)\mathrm{d}x = \left[x^3+x\right]_1^a = \left(a^3+a\right)-\left(1+1\right) = a^3+a-2$

$\int_1^b \left(6x^2+\frac{1}{x^2}\right)\mathrm{d}x = \left[2x^3-\frac{1}{x}\right]_1^b = \left(2b^3-\frac{1}{b}\right)-\left(2-1\right) = 2b^3-\frac{1}{b}-1$

11. $\int_1^k \left(1+\frac{1}{x^2}\right)\mathrm{d}x = \frac{3}{2}$ $\left[x-\frac{1}{x}\right]_1^k = \frac{3}{2}$

$k-\frac{1}{k}=\frac{3}{2} \Rightarrow 2k^2-3k-2=0 \quad \Rightarrow k=2$ since $k>1$

PROPERTIES OF DEFINITE INTEGRALS

12.

$\int_7^5 f(x)dx$	–2
$\int_5^7 3f(x)dx$	24
$\int_5^7 \frac{f(x)}{4}dx$	2
$\int_5^7 (f(x)+1)dx$	10
$\int_5^7 (f(x)+x)dx$	20
$\int_5^6 f(x)dx+\int_6^7 f(x)dx$	8
$\int_5^7 [f(x)+g(x)]dx$	10
$\int_5^7 [f(x)-g(x)]dx$	6
$\int_5^7 [2f(x)+3g(x)]dx$	22
$\int_5^7 [f(x)-4g(x)]dx$	0

13. (a) $\frac{1}{2}\times 10 = 5$

(b) $\int_1^3 g(x)\mathrm{d}x + \int_1^3 4\mathrm{d}x = 10 + [4x]_1^3 = 10+8=18$

14. (a) (i) $2\int_0^3 f(x)\,\mathrm{d}x = 16$

(ii) $\int_0^3 f(x)\,\mathrm{d}x + \int_0^3 2\,\mathrm{d}x$ =8+6= 14

(b) $\int_0^1 f(x)dx + \int_1^3 f(x)dx = \int_0^3 f(x)\,\mathrm{d}x = 8$

15. (a) 10

(b) $\int_1^3 3x^2+f(x)\mathrm{d}x = \int_1^3 3x^2\mathrm{d}x + \int_1^3 f(x)\mathrm{d}x$

$\int_1^3 3x^2\mathrm{d}x = \left[x^3\right]_1^3 = 27-1 = 26$

Thus $\int_1^3 3x^2+f(x)\mathrm{d}x = 26+5 = 31$

16. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(f(x)+g(x)\big)=f'(x)+g'(x)\ \ \big(=f'(4)+g'(4)\big)=7+4=11$

(b) $\int_1^3\big(g'(x)+6\big)\mathrm{d}x=\big[g(x)\big]_1^3+\big[6x\big]_1^3=\big(g(3)-g(1)\big)+\big(18-6\big)\ \big(=(2-1)+12\big)=13$

17. (a) $\int_1^5 3f(x)\mathrm{d}x=3\int_1^5 f(x)\mathrm{d}x=12$

Thus $\int_1^5 f(x)\,\mathrm{d}x=4$, and so $\int_5^1 f(x)\mathrm{d}x=-4$

(b) $I=\int_1^2\big(x+f(x)\big)\mathrm{d}x+\int_2^5\big(x+f(x)\big)\mathrm{d}x=\int_1^5\big(x+f(x)\big)\mathrm{d}x$

$\int_1^5 x\mathrm{d}x+\int_1^5 f(x)\mathrm{d}x=12+4=16$ (since $\int_1^5 x\mathrm{d}x=\left[\dfrac{x^2}{2}\right]_1^5=\dfrac{25}{2}-\dfrac{1}{2}=12$)

AREAS

18. (a) $\int_0^1 12x^2(1-x)\mathrm{d}x$

(b) $12\int_0^1(x^2-x^3)\,\mathrm{d}x=12\left[\dfrac{x^3}{3}-\dfrac{x^4}{4}\right]_0^1=12\left(\dfrac{1}{3}-\dfrac{1}{4}\right)=1$

19. (a) $\int_0^2(6x-3x^2)dx$

(b) 4

20. (a) $\int_0^2(-3x^2+8x-2x)dx=\int_0^2(6x-3x^2)dx$

(b) 4

21. (a) $\int_0^2 2xdx+\int_2^{8/3}(8x-3x^2)dx$

(b) $\dfrac{148}{27}$ ($\cong 5.48$)

22. (a) $\int_2^{8/3}(2+x-x^2)dx=4.5$

(b) $\int_2^{8/3}\big[(3+x-x^2)-(3-3x+x^2)\big]dx=2.67$

(c) Also 2.67

23. (a) At A, $x=0.753$ At B, $x=2.45$

(b) Area $\int_{0.753}^{2.45}y\mathrm{d}x$

(c) $=1.78$

24. 2.77

TRAPEZOIDAL RULE

25. (a) 4

(b) (i) $\frac{1}{2}h[y_0 + y_2 + 2y_1] = \frac{1}{2}\times 1\times[0+0+2\times 3] = 3$

(ii) $\frac{1}{2}h[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{1}{2}\times 0.5\times[0+0+2\times(\frac{9}{4}+3+\frac{9}{4})] = \frac{15}{4} = 3.75$

(iii) $\frac{1}{2}h[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$

$= \frac{1}{2}\times 0.4\times[0+0+2\times(1.92+2.88+2.88+1.92)] = 3.84$

26. (a)

x	1	1.5	2	2.5	3
$f(x)$	0	0.405	0.693	0.916	1.10

(b) (i) $\frac{1}{2}h[y_0 + y_2 + 2y_1] = \frac{1}{2}\times 1\times[0+1.10+2\times 0.693] = 1.243$

(ii) $\frac{1}{2}h[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$

$= \frac{1}{2}\times 0.5\times[0+1.10+2\times(0.405+0.693+0.916)] = 1.282$

(c) Percentage error = $\left|\frac{1.282-(3\ln 3-2)}{3\ln 3-2}\right|\times 100\% = 1.07\%$

B. Paper 2 questions (LONG)

27. (a) $f''(x) = 2x - 2 \Rightarrow f'(x) = x^2 - 2x + c$

When $x = 3, f'(x) = 0$: $0 = 9 - 6 + c \Rightarrow c = -3$

$f'(x) = x^2 - 2x - 3$

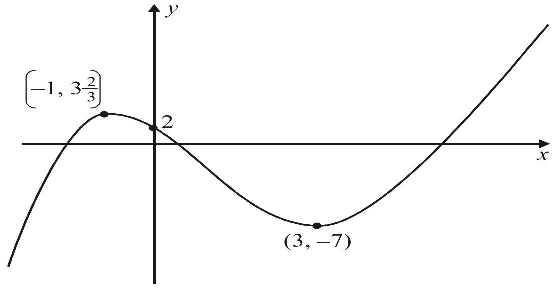
$f(x) = \frac{x^3}{3} - x^2 - 3x + d$

When $x = 3$, $f(x) = -7$: $-7 = 9 - 9 - 9 + d \Rightarrow d = 2$

$f(x) = \frac{x^3}{3} - x^2 - 3x + 2$

(b) $f(0) = 2, f(-1) = 11/3, f'(1) = 1 + 2 - 3 = 0$

(c) $f'(-1) = 0 \Rightarrow (-1, 11/3)$ is a stationary point



28. (a) $f'(2) = 0$

$6 - 2 + p = 0 \Leftrightarrow 4 + p = 0 \Leftrightarrow p = -4$

(b) integration/: $f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - 4x + c$

substituting (2, 4)

$\frac{1}{2}\times 2^3 - \frac{1}{2}\times 2^2 - 4\times 2 + c = 4,\ c = 10$

$f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - 4x + 10$

(c) $f'(x) = \frac{3}{2}x^2 - x - 4 = 0 \Leftrightarrow\ x = 2$ or $x = -\frac{4}{3}$

Hence the x-coordinate of B is $x = -\frac{4}{3}$

29. (a) $y = x(x - 4)^2$

$y = 0 \Leftrightarrow x = 0$ or $x = 4$

(b) $y = x(x - 4)^2 = x^3 - 8x^2 + 16x$

$\frac{dy}{dx} = 3x^2 - 16x + 16$

(c) $\frac{dy}{dx} = 3x^2 - 16x + 16$

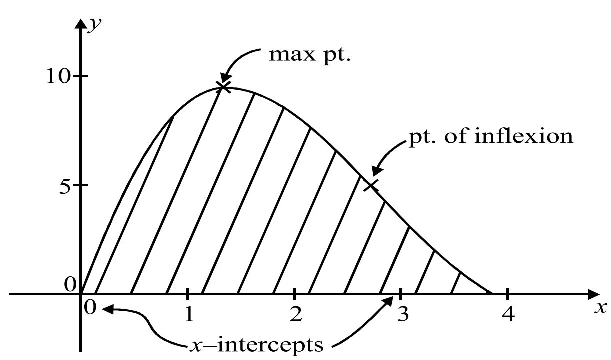
$\frac{dy}{dx} = 0 \Rightarrow x = 4$ or $x = \frac{4}{3}$

Using a table we see that

$x = \frac{4}{3}$ is a **maximum** $\Rightarrow y = \frac{256}{27}$ Thus $\left(\frac{4}{3}, \frac{256}{27}\right)$ (or (1.33, 9.48))

$x = 4$ is a **minimum** $\Rightarrow y = 0$ Thus (4, 0).

(d)



(e) (i) See diagram above

(ii) $0 < y < 10$ for $0 \le x \le 4$

So $\int_0^4 0\,dx < \int_0^4 y\,dx < \int_0^4 10\,dx \Rightarrow 0 < \int_0^4 y\,dx < 40$

30. (a) $x = -3, x = 0, x = 5$

(b) (i) $f'(x) = -3x^2 + 4x + 15$

$-3x^2 + 4x + 15 = 0 \Leftrightarrow x = -\frac{5}{3}$ or $x = 3$

(ii) $x = 3 \Rightarrow f(3) = -27 + 18 + 45 = 36$

(c) (i) At $x = 0, f'(0) = 15$
Line through (0, 0) of gradient 15 $\Rightarrow y = 15x$

(ii) $-x^3 + 2x^2 + 15x = 15x$ **by GDC** $x = 2$

(d) Area =115 (3 s.f.)

31. (a) (i) $f'(x) = -\frac{3}{2}x + 1$ $f'(2) = -2$, gradient of the normal $\left(\frac{1}{2}\right)$

$y - 3 = \frac{1}{2}(x - 2)$ $\left(\text{or } y = \frac{1}{2}x + 2\right)$

(ii) $-\frac{3}{4}x^2 + x + 4 = \frac{1}{2}x + 2 \Leftrightarrow 3x^2 - 2x - 8 = 0 \Leftrightarrow x = -\frac{4}{3}$ $(= -1.33)$

(b) (i) $\int_{-1}^{2}\left(-\frac{3}{4}x^2 + x + 4\right)\mathrm{d}x, \left[-\frac{1}{4}x^3 + \frac{1}{2}x^2 + 4x\right]_{-1}^{2}$

(ii) Area $= \frac{45}{4}(= 11.25)$ (accept 11.3)

32. (a) (i) $f'(x) = -x + 2$ (ii) $f'(0) = 2$

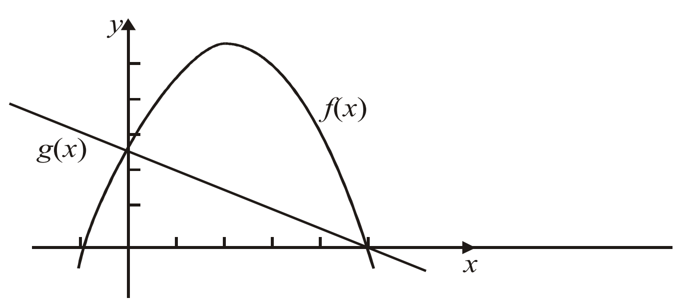
(b) y-intercept : At $x = 0, y = 2.5$

Gradient of tangent $= f'(0) = 2 \Rightarrow$ gradient of normal $= \frac{1}{2}$ $(= -0.5)$

the normal is $y - 2.5 = -0.5(x - 0) \Leftrightarrow (y = -0.5x + 2.5)$

(c) (i) **EITHER** solving $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Leftrightarrow x = 0$ or $x = 5$

OR



Curves intersect at $x = 0, x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$

Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point (5, 0)

(d) (i) Area $= \int_0^5 (f(x) - g(x))\mathrm{d}x = \int_0^5 (-0.5x^2 + 2x + 2.5)\mathrm{d}x - \frac{1}{2} \times 5 \times 2.5\Big)$

(the second integral is the area of triangle)

(ii) $A1 = \frac{50}{3}, A_2 = \frac{25}{4}$ Area $= \frac{50}{3} - \frac{25}{4} = \frac{125}{12}$ (or 10.4 (3s.f.)

33. (a) $f(1) = 2$

$f'(x) = 4x, \quad f'(1) = 4$

T: $y - 2 = 4(x - 1) \Rightarrow y = 4x - 2$

(b) $4x - 2 = 0 \Leftrightarrow x = \frac{1}{2}$

(c) **METHOD 1 (using only integrals)**

(i) Area $= \int_0^{0.5} 2x^2 \mathrm{d}x + \int_{0.5}^{1} f(x) - (4x - 2)\mathrm{d}x$

(ii) $\int 2x^2 \mathrm{d}x = \frac{2x^3}{3}, \int (4x - 2)\mathrm{d}x = 2x^2 - 2x$

Area $= \frac{1}{12} + \frac{2}{3} - 2 + 2 - \left(\frac{1}{12} - \frac{1}{2} + 1\right) = \frac{1}{6}$

METHOD 2 (using integral and triangle)

(i) Area $= \int_0^1 f(x)\mathrm{d}x -$ area of triangle $= \int_0^1 f(x)\mathrm{d}x - \frac{1}{2}$

(ii) $\int 2x^2 \mathrm{d}x = \frac{2x^3}{3}$

Area $= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

34. (a) limits $x = 0$, $x = 5$

area $= \int_0^5 f(x)\mathrm{d}x = 52.1$

(b) area is $\int_0^a x(a - x)\mathrm{d}x = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^3}{2} - \frac{a^3}{3}$

$\frac{a^3}{2} - \frac{a^3}{3} = 52.1 \Leftrightarrow a^3 = 6 \times 52.1 \Leftrightarrow a = 6.79$